Modularity of rigid Calabi–Yau 3–folds

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Effective Methods in Algebraic and Analytic Geometry

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Joint work with Duco van Straten and Christian Meyer (Mainz)

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[2] S. C., D. van Straten, *Infinitesimal deformations of double covers of smooth algerbaic varieties*, preprint math.AG/0303329.

[3] S. C., C. Meyer, *Geometry and Arithmetic of certain Double Octic Calabi-Yau Manifolds*, preprint math.AG/0304121, (to appear in Canadian Mathematical Bulletin)

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Equivalently: *Calabi–Yau* is a compact riemannian manifold with holonomy group in SU(3), *Ricci flat*.

(Calabi Conjecture (1954) proved by Yau (1976))

Numerical invariants of Calabi–Yau manifolds

- Euler characteristic $e(X)$,
- Hodge numbers $h^{i,j}$ $(0 \leq i, j \leq 3,\ i + i \leq 6)$,
- Betti numbers $b_0, \dots, b_6$, $(b_p = \sum_{i+j=p} h^{ij} = \dim_{\mathbb{C}} H^p(X, \mathbb{C}))$.

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Moreover the only non–trivial Hodge numbers have the following interpretation

$h^{1,1}$ $(=h^{2,2})$ equals the Picard number,

$h^{1,2}$ $(=h^{2,1})$ – number of deformations.

$l-$adic cohomology

Let X be a variety over an algebraic closed field of characteristic $p \geq 0$ and l a prime number $l \neq p$. There is a nice cohomology theory for X, the l*–adic cohomology*, which is a good replacement for complex cohomology

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- Poincare duality holds, i.e. there is a cup–product

$$H^i(X, \mathbb{Q}_l) \otimes \mathrm{H}^j(X, \mathbb{Q}_l) \longrightarrow \mathrm{H}^{i+j}(X, \mathbb{Q}_l),$$

$H^{2n}(X, \mathbb{Q}_l)$ is one dimensional and

$$H^i(X, \mathbb{Q}_l) \otimes \mathrm{H}^{2n-i}(X, \mathbb{Q}_l) \longrightarrow \mathrm{H}^{2n}(X, \mathbb{Q}_l)$$

is a perfect pairing.

Frobenius map

If X is a projective variety defined over a finite field $k = \mathbb{F}_p$ and $\bar{X} = X_{\mathrm{k}}\bar{\mathrm{k}}$ is the corresponding variety over an algebraic closure of k, we define a *Frobenius morphism*

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For a point $P \in \bar{X}$ we have $P \in \bar{X}_{p^r}$ iff P is a fixed point of Frob_p^r. If X is smooth then using the Lefschetz fixed–point formula we get

$$N_r := \# X_{p^r} = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(\mathrm{Frob}_p^{r*} : H^i(\bar{X}, \mathbb{Q}_l)).$$

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Example For $X = \mathbb{P}^n$ we have $Z(t) = \dfrac{1}{(1-t)(1-pt)\ldots(1-p^n t)}$.

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where E is the top Chern class of the tangent bundle of X.
- *Analogue of the Riemann hypothesis.* We can write
$$Z(t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots} P_{2n}(t),$$
where $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - p^n t$ and $P_i(t)$ is a polynomial with integer coefficients, $P_i(t) = \prod(1 - \alpha_{ij} t)$, where α_{ij} is an algebraic integer with $|\alpha_{ij}| = p^{i/2}$.

The proof of the Weil Conjecture (due to Deligne) is based on

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Taniyama–Shimura Conjecture: every elliptic curve E is *modular*, i.e. there exists a weight 2 and level N (where N is the conductor of E) cusp form such that the above coefficients a_p equal coefficient of the Fourier series of that modular form.

Modular forms.

We call $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ the *full modular group*. The group

$$\Gamma_o(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, c \equiv 0 (\mathrm{mod}\ N) \right\}, \text{ for } N \in \mathbb{N}$$

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An *unrestricted modular form* of weight k and level N is a holomorphic function f on the upper half plane $\mathbb{H}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_o(N), \quad \tau \in \mathbb{H}.$$

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f is called a *modular form* iff $c_n = 0$ for $n < 0$. If moreover $c_0 = 0$, it is a *cusp form*.

Modularity conjecture for Calabi–Yau manifolds

Let X be a Calabi–Yau manifold defined over $\mathbb{Q}$. Denote $t_i := \mathrm{tr}(\mathrm{Frob}_p^* : H^i(X, \mathbb{Q}_l))$. We have $t_0 = 1, t_1 = 0, t_5 = 0, t_6 = p^3$. By the Poincare duality $t_4 = pt_2$. Lefschetz fixed–point formula gives $\#X_p = 1 + p^3 + t_2(1 + p) - t_3$.

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Modularity conjectures for Calabi–Yau manifolds asserts that $a_p := t_3$ are coefficients of of a Fourier series of a modular form. There are special cases when this conjecture has a more explicit form (and some evidence for being true. One of them is when $b_2(X) = 2$. It is equivalent to X being rigid. Then a_p are conjectured to be coefficients of the Forier series of a weight 4 and level N cusp form, N being divisible only by the primes of bad reduction.

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Modularity conjecture for rigid Calabi–Yau manifold was almost proved by L. Dieulefait, J. Manoharmayum. Their proof do not determine the cusp form. There are only quite few known rigid Calabi–Yau manifolds, for all of them the cusp form was computed using a method due to Serre, Faltings and Livn (it is enough to verify the equality of a well-described finite set of coefficients.)

Double octic Calabi–Yau manifolds.

Let D be a sum of eight planes in $\mathbb{P}^3$, no six through a point, no four through a line. Then the double covering of $\mathbb{P}^3$ branched along D has a smooth model X being a Calabi–Yau manifold. There is a simple formula for the euler characteristc of X. There is also a method to compute the Hodge numbers.

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There are nine rigid examples in the described family (one has two variants). All examples have the following additional properties:

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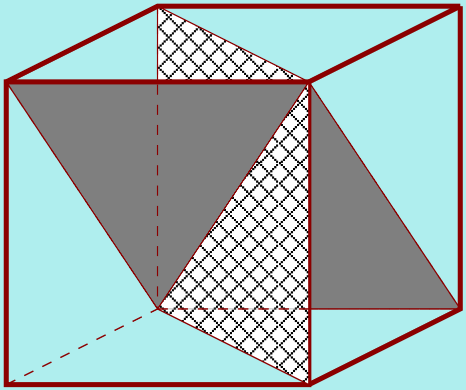
- their Picard groups are generated by divisors defined over $\mathbb{Q}$,
- there is a nice explicit description of a resolution of singularities (there are only eight types of singularities),
- the reduction mod p is smooth for all primes $p \geq 5$ (in fact for most of them reduction mod 3 is also smooth).

Arrangement no. 2

may be defined by the equation

$$xyzt(x+y)(y+z)(z+t)(t+x),$$

it consists of the faces of a tetrahedron and additional four planes going trough four vertices of the tetrahedron and intersecting in



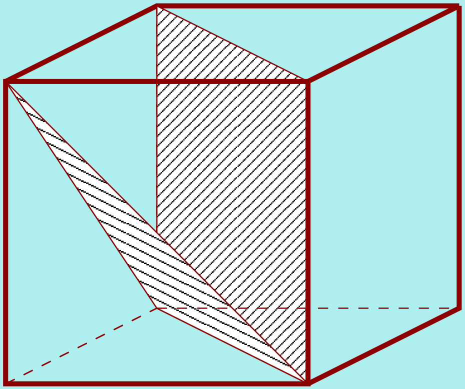
one point.

Arrangement no. 6

may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y-t)(x+y+z-t),$$

it consists of the faces of a cube and additional two planes, one through three vertices and the other through four vertices of the cube and intersecting along the diagonal of a face.

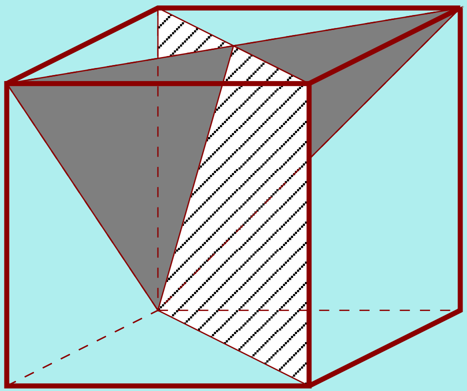


Arrangement no. 23

may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y-t)(x-y+z-t),$$

it consists of the faces of a cube and additional two planes, one through three vertices and the other through four vertices of the cube, having only one of the vertices of the cube in common.



Arrangement no. 29

may be defined by the equation

$$xyzt(x+y+z+t)(y+z+t)(x-z+t)(x+y+2t),$$

Arrangement no. 84

may be defined by the equation

$$xyzt(x+y+z+t)(2x+2z+t)(2y+2z+t)(x+y+2z+2t),$$

Sorry – no pictures

Arrangement no. 44

may be defined by the equation

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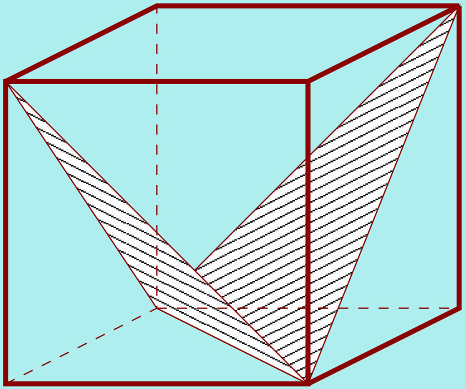
it consists of the faces of a cube and additional two planes through three vertices of the cube intersecting along the diagonal of a face.

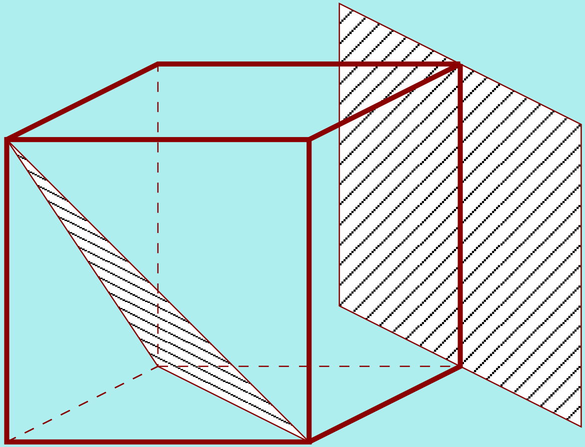


Arrangement no. 62

may be defined by the equation

$$xyz(x-t)(y-t)(z-t)(x+y+z-2t)(x+y),$$

it consists of the faces of a cube and additional two planes, one plane through an edge of the cube and parallel to a diagonal of the cube, and one plane through three vertices of the cube not belonging to the first plane.

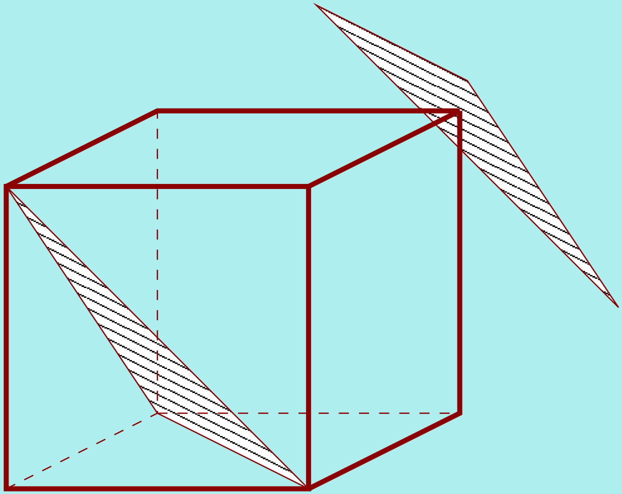


Arrangement no. 86

may be defined by the equation

$$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x+y+z+t)(x+y+z-3t),$$

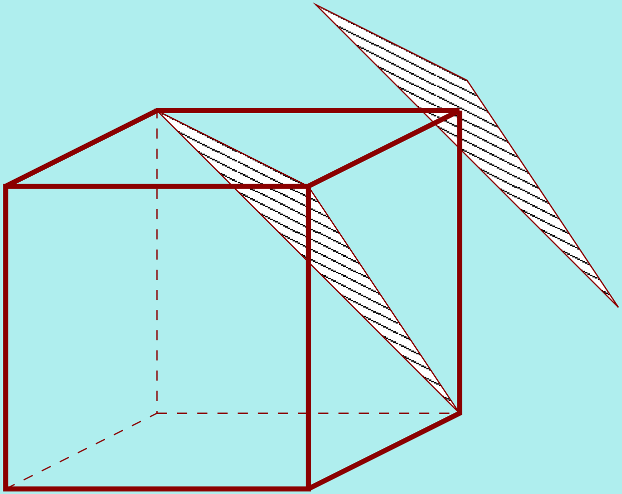
it consists of the faces of a cube and additional two parallel planes, one through three vertices of the cube and the second through one. The 4–fold points are: four vertices, three points at infinity which are the intersection of parallel edges of the cube, and three points of intersection at infinity of a pair of parallel faces of the cube and the additional two planes.



Arrangement no. 86^{a}

with the same numerical data as arrangement 86 may be defined by the equation

$$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x+y+z-t)(x+y+z-3t),$$

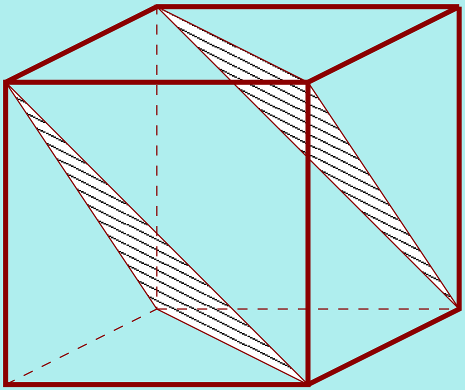


Arrangement no. 87

may be defined by the equation

$$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x+y+z+t)(x+y+z-t),$$

it consists of the faces of a cube and additional two parallel planes through three vertices. The 4–fold points are: six vertices, three points at infinity which are the intersection of parallel edges of the cube, and three points of intersection at infinity of a pair of parallel faces of the cube and the additional two planes.



Since the Picard groups are generated by divisors defined over $\mathbb{Q}$ we have $t_2 = ph^{1,1}$, $t_4 = p^2 h^{1,1}$ and so

$$a_p = 1 + p^3 + (p + p^2)h^{1,1} - \#\bar{X}_p.$$

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First we count points on the (*singular*) double covering of $\mathbb{P}^3$ – if the equation of the branch locus if $f(x, y, z, t) = 0$, we count points on $\mathbb{P}^3(\mathbb{F}_p)$ for which value of f is zero, square or not-square mod p. It is very easy (for a computer).

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Now we have to take into account resolution of singularities, for seven types of singularities it is very easy the number of added points depends only on the number of those types of singularities. Unfortunately if four planes intersects, but no three of them contains a line, blow–up of that point replace one point by a double covering of $\mathbb{P}^2$ branched along an arrangement of four lines. We count the points in a similar manner as before.

A C++ program gave the following table.

p	5	7	11	13	17	19	23	73
Arrangements 2, 87								
a_p	-2	24	-44	22	50	44	-56	154
Arrangement 6								
a_p	-10	-16	40	-50	-30	-40	-48	-630
Arrangement 23								
a_p	-22	0	0	18	-94	0	0	1098
Arrangements 29, 44								
a_p	-2	-24	44	22	50	-44	56	154
Arrangement 62								
a_p	2	-24	-44	-22	50	44	56	154
Arrangements 84, 86								
a_p	6	-16	12	38	-126	20	168	218
Arrangement 86[a]								
a_p	-18	8	36	-10	18	-100	72	26

Comparing above with the database of modular form (by W. Stein) we can recognize the following modular forms

A C++ program gave the following table.

p	5	7	11	13	17	19	23	73
Arrangements 2, 87					8k4A			
a_p	-2	24	-44	22	50	44	-56	154
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Arrangement 6					32k4C			
a_p	-10	-16	40	-50	-30	-40	-48	-630
Arrangement 23								
a_p	-22	0	0	18	-94	0	0	1098
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Arrangement 23					64k4A			
a_p	-22	0	0	18	-94	0	0	1098
Arrangements 29, 44								
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a_p	2	-24	-44	-22	50	44	56	154
Arrangements 84, 86					6k4A			
a_p	6	-16	12	38	-126	20	168	218
Arrangement 86[a]					12k4A			
a_p	-18	8	36	-10	18	-100	72	26

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