Modularity Conjecture for Calabi–Yau manifolds

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joint works with D. van Straten, Ch. Meyer, K. Hulek

Calabi–Yau manifolds

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Calabi–Yau manifold is a complex, projective (kähler) threefold X satisfying

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Another definition:
Calabi–Yau manifold is a compact riemannian manifold with holonomy group in SU(3) (holonomy group equal SU(3)).

Numerical invariants of Calabi–Yau manifolds

- Euler characteristic $e(X)$,
- Hodge numbers $h^{i,j} = \dim H^j(\Omega^i_X)$ $(0 \leq i, j \leq 3,\ i + i \leq 6)$,
- Betti numbers $b_0, \dots, b_6$, $(b_i = \dim_{\mathbb{C}} H^i(X, \mathbb{C}) = \sum_{p+q=i} h^{ij})$.

$$
\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{01} & & h^{10} & & \\
 & h^{02} & & h^{11} & & h^{20} & \\
h^{03} & & h^{12} & & h^{21} & & h^{30} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
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$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{12} & & h^{21} & & 1 \\ & 0 & & h^{22} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

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$e(X) = 2(h^{11} - h^{12})$
$h^{11} = \rho(X)$ Picard number
h^{12} dimension of deformation space

L–series of a Calabi–Yau manifold

Let X be a Calabi–Yau manifold defined over $\mathbb{Q}$ and let p be a prime of good reduction, i.e. the reduction X_p of X mod p is smooth. By the Weil Conjecture the Zeta function of X_p can be written as

$$\frac{P_{1,p}(t)P_{3,p}(t)P_{5,p}(t)}{P_{0,p}(t)P_{2,p}(t)P_{4,p}(t)P_{6,p}(t)}$$

where $P_{i,p}$ is a polynomial of degree b_i.

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where $P_{i,p}$ is a polynomial of degree b_i. We define the i–th cohomological L–series of X as

$$L(H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l),s) = (*) \prod_{p \text{ good prime}} \frac{1}{P_{i,p}(p^{-s})}$$

where $(*)$ stands for the Euler factors corresponding to the primes of bad reduction.

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where $(*)$ stands for the Euler factors corresponding to the primes of bad reduction. The most interesting is the third L–series

$$L(X, s) = L(H^3_{\text{ét}}(\bar{X}, \mathbb{Q}_l), s).$$

Frobenius morphism

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By the proof of Weil Conjecture

$$P_{i,p}(t)=det(1-t\,\mathrm{Frob}_p^*\,|H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l))$$

and so

$$a_p(X)=\mathrm{tr}(\mathrm{Frob}_p^*\,|H^i_{\text{ét}})$$

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Lefschetz fixed point formula

$$\#X_{p^r} = \sum_{i=0}^{6} (-1)^i\,\mathrm{tr}(\mathrm{Frob}_p^*\,|H^i_{\text{ét}}) = 1 + p^3 + \mathrm{tr}(\mathrm{Frob}_p^*\,|H^2_{\text{ét}})(1+p) - a_p$$

Modular forms

We call $\Gamma := \mathrm{SL}(2,\mathbb{Z})$ the *full modular group*. The group

$$\Gamma_o(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, c \equiv 0 (\mathrm{mod}\ N) \right\}, \text{ for } N \in \mathbb{N}$$

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An *unrestricted modular form* of weight k and level N is a holomorphic function f on the upper half plane $\mathbb{H}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_o(N), \quad \tau \in \mathbb{H}.$$

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If moreover $c_0 = 0$, it is a *cusp form*.

Modularity Conjecture for Calabi–Yau manifolds

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Two dimensional Calabi–Yau manifolds are K3 surfaces, in this case middle cohomology has rank 20, but we consider only its sublattice of transcendental cycles. In the case of maximal Picard number (singular K3 surfaces) Livné showed that the L–series is the one of some weight 3 modular form.

Modularity Conjecture has particulary explicit form for *rigid Calabi–Yau manifolds*, i.e. Calabi–Yau manifolds with $b_3 = 2$.

Conjecture (Modularity Conjecture for rigid Calabi–Yau manifolds)

Let X be a rigid Calabi–Yau manifold defined over $\mathbb{Q}$. Then there exists a weight four cusp form f for $\Gamma_0(N)$ such that $L(f, s) = L(X, s)$, where N is an integer divisible only by bad primes of X.

J.-M. Fontaine and B. Mazur *Geometric Galois representations*, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.

N. Yui, *Update on the modularity of Calabi-Yau varieties*, With an appendix by Helena Verrill. Fields Inst. Commun., 38, Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), 307–362, Amer. Math. Soc., Providence, RI, 2003.

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Modularity Conjecture for rigid Calabi–Yau manifold was proved by Dieulefait and Manoharmayum under very mild assumptions on the primes of bad reduction

L. Dieulefait and J. Manoharmayum, *Modularity of rigid Calabi-Yau threefolds over $\mathbb{Q}$*. Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), 159–166, Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003.

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Fiber product of elliptic fibrations (Schoen):
Let $p_1 : S_1 \longrightarrow \mathbb{P}^1$ and $p_2 : S_2 \longrightarrow \mathbb{P}^1$ be rational elliptic surfaces with sections, the fiber product $X = S_1 \times_{\mathbb{P}^1} S_2$ is a singular Calabi–Yau manifold. Singularities of X are located on the product of singular fibers of S_1 and S_2. If both fibers are semistable then the only singularities are nodes.

First examples of (modular) rigid Calabi–Yau manifolds

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The easiest case is of a self fiber product ($S_1 = S_2 = S$) and S semistable. Then X has a smooth model $\hat{X}$ defined over $\mathbb{Q}$ which is a calabi–Yau manifold with $h^{12}(\hat{X})$ equals to four minus number of singular fibers.

Self fiber products of Beauville surfaces

There are 6 semistable, rational elliptic surfaces with four singular fibers (Beauville surfaces). They give six modular Calabi–Yau threefolds

Γ	singular fibers	weight 4 form	level
$\Gamma(3)$	I_3, I_3, I_3, I_3	$\eta(3\tau)^8$	9
$\Gamma_1(4) \cap \Gamma(2)$	I_4, I_4, I_2, I_2	$\eta(2\tau)^4\eta(4\tau)^4$	8
$\Gamma_1(5)$	I_5, I_5, I_1, I_1	$\eta(\tau)^4\eta(5\tau)^4$	5
$\Gamma_1(6)$	I_6, I_3, I_2, I_1	$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$	6
$\Gamma_0(8) \cap \Gamma_1(4)$	I_8, I_2, I_1, I_1	$\eta(4\tau)^{16}\eta(2\tau)^{-4}\eta(8\tau)^{-4}$	16
$\Gamma_0(9) \cap \Gamma_1(3)$	I_6, I_3, I_2, I_1	$\eta(3\tau)^8$	8

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1 - q^n)$

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There are several proofs of modularity, one based on the Dieulefait–Manoharmayum theorem another on the Shimura isomorphism and based on the Faltings–Serre–Livné method.

The Faltings–Serre–Livné method

Theorem

Let S be a finite set of primes and let $\rho_1, \rho_2 : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gl}_2(\mathbb{Q}_2)$ *be continuous Galois representations, unramified outside S and satisfying*

1. $\operatorname{tr}\rho_1 = \operatorname{tr}\rho_2 = 0 \mod 2$ *and* $\det\rho_1 = \det\rho_2 \mod 2$
2. *There exists a finite set T of primes, disjoint from S for which the image of the set* $\{\mathrm{Frob}_t : \ t \in T\}$ *in* $\mathrm{Gal}(\mathbb{Q}_S/Q)$ *is non–cubic and*

$$\operatorname{tr}\rho_1(\mathrm{Frob}_t) = \operatorname{tr}\rho_2(\mathrm{Frob}_t) \text{ and } \det\rho_1(\mathrm{Frob}_t) = \det\rho_2(\mathrm{Frob}_t)$$

   for all $t \in T$.

Then ρ_1 and ρ_2 have isomorphic semi–simplifications. In particular $\operatorname{tr}\rho_1(\mathrm{Frob}_t) = \operatorname{tr}\rho_2(\mathrm{Frob}_t)$ *for all $t \notin S$.*

Faltings–Serre-Livné method

Corollary

Let X be a rigid Calabi–Yau manifold defined over $\mathbb{Q}$ and let

$$f(q) = \sum_{m=1}^{\infty} b_m q^m$$

be a cusp form of weight 4 for $\Gamma_o(N)$.Let S be a set of primes containning bad primes for X and prime divisors of N. Suppose that

$$a_p = b_p = 0 \mod 2$$

for all primes $p \notin S$ and $a_p(X) = b_p$for all $p \in T$. Then $L(X, s) = L(f, s)$, in particular $a_p(X) = b_p$ for $p \not]inS$.

The set $T$is easy to describe for a given S.

Double octic Calabi–yau manifolds

Let $D \subset \mathbb{P}^3$ be a smooth octic surface, $\pi : X \longrightarrow \mathbb{P}^3$ double covering branched along D. Then X is a Calabi–yau manifold. If D is singular then X is singular as well (singulrities of X coresponde to those of D). However, D are "nice" there can exists a Calabi–yau resolution of singularities.

I have studied the case of D being an octic arrangement (locally looks like intersection of plane) with no sixfold point nor a fourfold curve. Then we have a nice description of resolution of singularities and formula for Hodge numbers. Also using the Lefschetz fixed point formula it is easy to compute the trace of Frobenius hence to prove modularity in rigid cases.

There are two explicit formulations for modularity of non–rigid Calabi–Yau manifolds. One is (due to Livné and Yui)

$$L(X,s) = L(f_2 \otimes f_3) \text{ or } L(X,s) = L(f_2^1 \otimes f_2^2 \otimes f_2^3).$$

Examples come from a singular K3 surface S with involution i_S and elliptic curve E with involution i_E. Then on $E \times S$ we have diagonal involution i and the quotient $(E \times S)/i$ has a Calabi–Yau resolution of singularities.

Similarly, if E_j is an elliptic curve with involution i_j $(j = 1, 2, 3)$, then $(E_1 \times E_2 \times E_3)/D_2$ (where $D_2 = \{1, (i_1, i_2, 1), (i_1, 1, i_3), (1, i_2, i_3)\}$) has a Calabi–Yau resolution of singularities.

The Calabi–Yau manifolds have L–series as above.

Modularity of non–rigid Calabi–Yau manifolds

Another formularion for modularity was suggested by Hulek and Verrill. Let $S_i, i = 1, \dots, r$ $(r = h^{21}(X))$ be a birational ruled surface over an elliptic curve E_i. Assume that the map

$$H^3(X, \mathbb{C}) \longrightarrow \bigoplus_{i=1}^{r} H^3(S_i, \mathbb{C})$$

is surjective.
Under the same assumptions on bad primes as in Dieulefait and Manoharmayum we get

$$L(X, s) = L(g_4, s) \prod_j L(g_2^j, s-1)$$

where g_2^i is the weiight 2 modular form for elliptic curve E_i and g_4 is some weight 4 modular form.

If $L(X,s) = L(g_4,s)\prod_j L(g_2^j, s-1)$ then

$$a_p(X) = b_p + p(c_p^1 + \cdots + c_p^r),$$

where b_p (resp. c_p^j) is a Fourier coefficient of g_4 (resp. g_2^j). In particular $a_p(X) \cong b_p \mod p$.

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C. Meyer run an enormous computer search for double octic Calabi–Yau manifolds with $h^{12} = 0$ or 1 satisfying the above "numerical evidence" of modularity. Modularity of all 11 rigid examples can be proved using the Faltings–Serre–Livné method, the only delicate point is to correctly count points added in resolution of singularities.

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In fact all proofs were based on finding a correspondence with a fiber product of elliptic fibrations, all studied examples were given by K3 fibrations with large Picard number, a K3 surface with large Picard number is related to a product of elliptic curves.

Corespondences between modular Calabi–Yau manifolds

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We were able to construct only a few examples with some ad hoc methods, in most cases we proceed by constructing first correspondences with some fibers products of rational elliptic fibrations.

Higher dimensions

The first known modular example in dimension bigger then 3 was due to Ahlgren. Let X be the variety given in $\mathbb{C}^6$ by

$$w^2 = x(x-1)(x-\lambda)y(y-1)(y-\lambda)z(z-1)(z-\lambda)t(t-1)(t- lambda)$$

Theorem (Ahlgren)

$$\#X_p = p^5 + 2p^3 - 4p^2 - 9p - 1 - b_p,$$

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With K. Hulek we proved that the projective closure (which is a double cover of $\mathbb{P}^5$ branched along twelve hyperplanes) has a Calabi–Yau smooth model $\tilde{X}$ with $b_5(\tilde{X}) = 2$ and $a_p(\tilde{X}) = b_p$.

Let E (resp. F) be the elliptic curve given by the Weierstrass–ype equation

$$y^2 = x^3 - D \text{ (resp. } y^2 = x^3 - x).$$

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Then E has action of $\mathbb{Z}_3$ and F has an action of $\mathbb{Z}_4$.
The quotients $X^n = E^n/\mathbb{Z}_3^{n-1}$ and $Y^n = F^n/\mathbb{Z}_4^{n-1}$ have Calabi–Yau resolutions of singularities $\tilde{Y}^n$such that for n odd $b_n(\tilde{X}^n) = b_n(\tilde{Y}_n) = 2$ and the L–series of $\tilde{X}^n$ (resp. $\tilde{Y}^n$) equals the L–series of the modular form corresponding to the n–th power of the Hecke character corresponding to the curve E (resp. F).